Mathematica Balkanica

New Series Vol. 27, 2013, Fasc. 1–2

On the Mixed Modulus of Smoothness and a Class of Double Fourier Series

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Presented by Blagovest Sendov

In this paper we have defined a new class of double numerical sequences. If the coefficients of a double cosine or sine trigonometric series belong to the such classes, then it is verified that they are Fourier series or equivalently their sums are integrable functions. In addition, we obtain an estimate for the mixed modulus of smoothness of a double sine Fourier series whose coefficients belong to the new class of sequences mention above.

MSC 2010: 42A32; 42A20
Key Words: double Fourier series, modulus of smoothness, Fejér's kernel

1. Known results on single trigonometric series

Let us consider the trigonometric series

$$\sum_{k=1}^{\infty} a_k \cos kx \tag{1}$$

and

$$\sum_{k=1}^{\infty} b_k \sin kx, \tag{2}$$

whose coefficients tend to zero, in other words

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = 0.$$

An interesting topic that concerns with above series is the so-called *Fourier Series Problem* and *Integrability Problem*, which are equivalent problems. These problems consist of finding the properties of the coefficients a_k, b_k such that the above series are Fourier series.

In most cases cosine series (1) is more prickly than sine series (2). For instance, when $b_k \downarrow 0$ then (2) is a Fourier series if and only if

$$\sum_{k=1}^{\infty} \frac{b_k}{k} < \infty,$$

but for the series (1) with $a_k \downarrow 0$, the condition

$$\sum_{k=1}^{\infty} \frac{a_k}{k} < \infty$$

is only sufficient one.

Another well-known sufficient condition due to W. H. Young is that $a_k \to 0$ and $\{a_k\}$ is quasi-convex. Regarding to the necessity condition we can mention the result of R. Salem (see [1], Vol. 1, page 237) that

$$(a_k - a_{k+1}) \log k \to 0$$

when $a_k \downarrow 0$. Also, it is well-known that when coefficients of the series (1) and (2) tend to zero and are of bounded variation, then they are Fourier series if and only if they represent integrable functions. On the other hand, S. A. Telyakovskii [5] has verified that when $b_k \to 0$ and $\{b_k\}$ is quasi-convex, then (2) represents an integrable function if and only if

$$\sum_{k=1}^{\infty} \frac{|b_k|}{k} < \infty.$$

Later, on one hand T. Kano [3] has proved the following theorem.

Theorem 1.1. *If $\{c_k\}$, ($c_k = a_k$ or b_k) is a null sequence such that*

$$\sum_{k=1}^{\infty} k^2 \left| \triangle^2 \left(\frac{c_k}{k} \right) \right| < +\infty, \tag{3}$$

then (1) and (2) are Fourier series, or equivalently, they represent integrable functions.

On the other hand B. Ram and S. Kumari [4] have used the class of null sequences that satisfy condition (3) (we denote it by $\mathcal{R}$) in order to estimate the integral modulus of continuity of a function in terms of its Fourier coefficients.

Let $G \in L^p$ $(1 \le p < \infty)$ be a $2\pi-$periodic function. The integral modulus of continuity of order m of G in L^p is defined by

$$\omega_p^m(h; G) = \sup_{0<|t|\le h} \|\triangle_t^h G(x)\|_{L^p},$$

where

$$\triangle_t^h G(x) = \sum_{\alpha=0}^{m} (-1)^{m-\alpha} \binom{m}{\alpha} G(x + \alpha t),$$

and $\|(\cdot)\|_{L^p}$ denotes the norm in L^p.

Estimating the integral modulus of continuity of a function in terms of its Fourier coefficients has a "long history". For example, M. Izumi and S. Izumi have proved an estimate of the integral modulus of continuity of order 1 of a function whose Fourier coefficients are quasi-convex (see [2]). An improvement of their result is given by S. A. Telyakovskiĭ [5], and another studying in this topic is one interesting result of B. Ram and S. Kumari [4] formulated below.

Theorem 1.2. *Let $\{a_k\} \in \mathcal{R}$ be a null-sequence. Then*

$$\omega_1^m\left(\frac{1}{n}; g\right) \le \frac{A_m}{n^m} \sum_{k=1}^{n} (k+1)^{m+2} \left|\triangle^2\left(\frac{b_k}{k}\right)\right| + A_m \sum_{k=n+1}^{\infty} (k+1)^2 \left|\triangle^2\left(\frac{b_k}{k}\right)\right|,$$

where $g(x)$ is the sum function of the series (2).

Our aim in this paper is to extend Theorem 1.1 and Theorem 1.2 from single to double Fourier series. In order to prove the main results we need the following auxiliary statement [4].

Lemma 1.3. *Let $0 < t \le 1/n$, $(n = 1, 2, \dots)$, and let m be a natural number. If $K_\nu(x)$ denotes the Fejér's kernel, then*

$$\int_0^\pi \left|\triangle_{\pm t}^m K_\nu'(x)\right| dx \le \begin{cases} A_m t^m \nu^{m+1} & \text{for } \nu = 1, 2, \dots, n \\ A_m \nu & \text{for } \nu = 1, 2, \dots . \end{cases}$$

Throughout this paper $O-$symbol contain positive constants, generally speaking, different in different estimates.

2. New results on double trigonometric series

Let us consider the trigonometric series

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} \cos mx \cos ny \tag{4}$$

and

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{m,n}\sin mx \sin ny, \tag{5}$$

whose coefficients satisfy condition

$$a_{m,n},\ \ b_{m,n}\to 0 \quad \text{as} \quad m+n\to\infty.$$

The following notations will be used throughout the paper. For an arbitrary numerical sequence $u_{m,n}$ we denote

$$\begin{aligned}
\triangle_{10}u_{\nu,\mu} &= u_{\nu,\mu}-u_{\nu+1,\mu}, \qquad \triangle_{01}u_{\nu,\mu}=u_{\nu,\mu}-u_{\nu,\mu+1},\\
\triangle_{11}u_{\nu,\mu} &= \triangle_{10}\left(\triangle_{01}u_{\nu,\mu}\right), \quad \triangle_{21}u_{\nu,\mu}=\triangle_{10}\left(\triangle_{11}u_{\nu,\mu}\right),\\
\triangle_{12}u_{\nu,\mu} &= \triangle_{01}\left(\triangle_{11}u_{\nu,\mu}\right), \quad \triangle_{22}u_{\nu,\mu}=\triangle_{11}\left(\triangle_{11}u_{\nu,\mu}\right).
\end{aligned}$$

Now we are able to give the following definition.

Definition 2.1. *We say that $\{u_{m,n}\}$ belongs to the class $\mathcal{R}^2$ if $u_{m,n}\to 0$ as $m+n\to\infty$, and*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^2\left|\triangle_{22}\left(\frac{u_{m,n}}{mn}\right)\right|<\infty.$$

First, we prove the following result.

Theorem 2.2. *If $\{c_{m,n}\}\in\mathcal{R}^2$ (where $c_{m,n}$ is $a_{m,n}$ or $b_{m,n}$), then the series (4) and (5) are Fourier series, or equivalently, they represent integrable functions.*

P r o o f. We shall prove the theorem only for the series (5) since for the series (4) it can be proved in a very same way. Applying the summation by parts we have

$$
\begin{aligned}
S_{m,n}(x,y) &= \sum_{k=1}^{m}\sum_{\ell=1}^{n} b_{k,\ell}\sin kx \sin \ell y \\
&= \frac{\partial^2}{\partial x \partial y}\sum_{k=1}^{m}\sum_{\ell=1}^{n}\frac{b_{k,\ell}}{k\ell}\cos kx \cos \ell y \\
&= \frac{\partial}{\partial x}\sum_{k=1}^{m}\cos kx\left[\sum_{\ell=1}^{n-1}\triangle_{01}\left(\frac{b_{k,\ell}}{k\ell}\right)D'_\ell(y)+\frac{b_{k,n}}{kn}D'_n(y)\right] \\
&= \frac{\partial}{\partial x}\sum_{k=1}^{m}\cos kx\left[\sum_{\ell=1}^{n-2}(\ell+1)\triangle_{02}\left(\frac{b_{k,\ell}}{k\ell}\right)K'_\ell(y)\right. \\
&\quad \left. +n\triangle_{01}\left(\frac{b_{k,n-1}}{k(n-1)}\right)K'_{n-1}(y)+\frac{b_{k,n}}{kn}D'_n(y)\right] \\
&= \sum_{\ell=1}^{n-2}(\ell+1)K'_\ell(y)\frac{\partial}{\partial x}\sum_{k=1}^{m}\triangle_{02}\left(\frac{b_{k,\ell}}{k\ell}\right)\cos kx \\
&\quad +nK'_{n-1}(y)\frac{\partial}{\partial x}\sum_{k=1}^{m}\triangle_{01}\left(\frac{b_{k,n-1}}{k(n-1)}\right)\cos kx + D'_n(y)\frac{\partial}{\partial x}\sum_{k=1}^{m}\frac{b_{k,n}}{kn}\cos kx \\
&= \sum_{\ell=1}^{n-2}(\ell+1)K'_\ell(y)\left[\sum_{k=1}^{m-1}\triangle_{12}\left(\frac{b_{k,\ell}}{k\ell}\right)D'_k(x)+\triangle_{02}\left(\frac{b_{m,\ell}}{m\ell}\right)D'_m(x)\right] \\
&\quad +nK'_{n-1}(y)\left[\sum_{k=1}^{m-1}\triangle_{11}\left(\frac{b_{k,n-1}}{k(n-1)}\right)D'_k(x)+\triangle_{01}\left(\frac{b_{m,n-1}}{m(n-1)}\right)D'_m(x)\right] \\
&\quad +D'_n(y)\left[\sum_{k=1}^{m-1}\triangle_{10}\left(\frac{b_{k,n}}{kn}\right)D'_k(x)+\frac{b_{m,n}}{mn}D'_m(x)\right] \\
&= \sum_{\ell=1}^{n-2}(\ell+1)K'_\ell(y)\left[\sum_{k=1}^{m-2}(k+1)\triangle_{22}\left(\frac{b_{k,\ell}}{k\ell}\right)F'_k(x)\right. \\
&\quad \left. +m\triangle_{12}\left(\frac{b_{m-1,\ell}}{(m-1)\ell}\right)F'_{m-1}(x)+\triangle_{02}\left(\frac{b_{m,\ell}}{m\ell}\right)D'_m(x)\right] \\
&\quad +nK'_{n-1}(y)\left[\sum_{k=1}^{m-2}(k+1)\triangle_{21}\left(\frac{b_{k,n-1}}{k(n-1)}\right)F'_k(x)\right. \\
&\quad \left. +m\triangle_{11}\left(\frac{b_{m-1,n-1}}{(m-1)(n-1)}\right)F'_{m-1}(x)+\triangle_{01}\left(\frac{b_{m,n-1}}{m(n-1)}\right)D'_m(x)\right]+
\end{aligned}
$$

$$
\begin{aligned}
&+D'_n(y)\Bigg[\sum_{k=1}^{m-2}(k+1)\triangle_{20}\left(\frac{b_{k,n}}{kn}\right)F'_k(x)\\
&+m\triangle_{10}\left(\frac{b_{m-1,n}}{(m-1)n}\right)F'_{m-1}(x)+\frac{b_{m,n}}{mn}D'_m(x)\Bigg]\\
= &\sum_{k=1}^{m-2}\sum_{\ell=1}^{n-2}(k+1)(\ell+1)\triangle_{22}\left(\frac{b_{k,\ell}}{k\ell}\right)F'_k(x)K'_\ell(y)\\
&+\sum_{k=1}^{m-2}(k+1)n\triangle_{21}\left(\frac{b_{k,n-1}}{k(n-1)}\right)F'_k(x)K'_{n-1}(y)\\
&+\sum_{k=1}^{m-2}(k+1)\triangle_{20}\left(\frac{b_{k,n}}{kn}\right)F'_k(x)D'_n(y)\\
&+\sum_{\ell=1}^{n-2}m(\ell+1)\triangle_{12}\left(\frac{b_{m-1,\ell}}{(m-1)\ell}\right)F'_{m-1}(x)K'_\ell(y)\\
&+\sum_{\ell=1}^{n-2}(\ell+1)\triangle_{02}\left(\frac{b_{m,\ell}}{m\ell}\right)D'_m(x)K'_\ell(y)\\
&+mn\triangle_{11}\left(\frac{b_{m-1,n-1}}{(m-1)(n-1)}\right)F'_{m-1}(x)K'_{n-1}(y)\\
&+m\triangle_{10}\left(\frac{b_{m-1,n}}{(m-1)n}\right)F'_{m-1}(x)D'_n(y)\\
&+n\triangle_{01}\left(\frac{b_{m,n-1}}{m(n-1)}\right)D'_m(x)K'_{n-1}(y)\\
&+\frac{b_{m,n}}{mn}D'_m(x)D'_n(y),
\end{aligned}
$$

where $K_p(v)$ denotes the Fejér's kernel defined by the equality

$$K_p(v)=\frac{1}{p+1}\sum_{j=0}^{p}D_j(v).$$

Using the estimation $|D'_p(v)|=O(p)$ we obtain

$$|K'_p(v)|=O(p),$$

therefore we have

$$\begin{aligned}
&\sum_{k=1}^{m-2}(k+1)n\left|\triangle_{21}\left(\frac{b_{k,n-1}}{k(n-1)}\right)\right||F'_k(x)||K'_{n-1}(y)| \\
&\qquad = O\left(n(n-1)\sum_{k=1}^{m-2}k^2\left|\sum_{\ell=n-1}^{\infty}\triangle_{22}\left(\frac{b_{k,\ell}}{k\ell}\right)\right|\right) \\
&\qquad = O\left(\sum_{k=1}^{m-2}\sum_{\ell=n-1}^{\infty}(k\ell)^2\left|\triangle_{22}\left(\frac{b_{k,\ell}}{k\ell}\right)\right|\right)\to 0,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=1}^{m-2}(k+1)\left|\triangle_{2,0}\left(\frac{b_{k,n}}{kn}\right)\right||F'_k(x)||D'_n(y)| \\
&\qquad = O\left(n\sum_{k=1}^{m-2}k^2\left|\sum_{\ell=n}^{\infty}\triangle_{21}\left(\frac{b_{k,\ell}}{k\ell}\right)\right|\right) \\
&\qquad = O\left(\sum_{k=1}^{m-2}k^2\sum_{\ell=n}^{\infty}\ell\left|\triangle_{21}\left(\frac{b_{k,\ell}}{k\ell}\right)\right|\right) \\
&\qquad = O\left(\sum_{k=1}^{m-2}k^2\sum_{\ell=n}^{\infty}\sum_{i=\ell}^{\infty}\ell\left|\triangle_{22}\left(\frac{b_{k,i}}{ki}\right)\right|\right) \\
&\qquad = O\left(\sum_{k=1}^{m-2}k^2\sum_{\ell=n}^{\infty}\left(\sum_{i=n}^{\ell}i\right)\left|\triangle_{22}\left(\frac{b_{k,\ell}}{k\ell}\right)\right|\right) \\
&\qquad = O\left(\sum_{k=1}^{m-2}k^2\sum_{\ell=n}^{\infty}\left(\sum_{i=1}^{\ell}i\right)\left|\triangle_{22}\left(\frac{b_{k,\ell}}{k\ell}\right)\right|\right) \\
&\qquad = O\left(\sum_{k=1}^{m-2}\sum_{\ell=n}^{\infty}(k\ell)^2\left|\triangle_{22}\left(\frac{b_{k,\ell}}{k\ell}\right)\right|\right)\to 0,
\end{aligned}$$

as $n\to\infty$ uniformly in m.

Similarly,

$$\sum_{\ell=1}^{n-2}m(\ell+1)\left|\triangle_{12}\left(\frac{b_{m-1,\ell}}{(m-1)\ell}\right)\right||F'_{m-1}(x)||K'_\ell(y)|\to 0$$

and

$$\sum_{\ell=1}^{n-2}(\ell+1)\left|\triangle_{02}\left(\frac{b_{m,\ell}}{m\ell}\right)\right||D'_m(x)||K'_\ell(y)|\to 0$$

as $m\to\infty$ uniformly in n.

Also, we have

$$\begin{aligned} mn\left|\triangle_{11}\left(\frac{b_{m-1,n-1}}{(m-1)(n-1)}\right)\right| &|F'_{m-1}(x)||K'_{n-1}(y)| \\ &= O\left(m(m-1)n(n-1)\sum_{k=m-1}^{\infty}\sum_{\ell=n-1}^{\infty}\left|\triangle_{22}\left(\frac{b_{k,\ell}}{k\ell}\right)\right|\right) \\ &= O\left(\sum_{k=m-1}^{\infty}\sum_{\ell=n-1}^{\infty}(k\ell)^2\left|\triangle_{22}\left(\frac{b_{k,\ell}}{k\ell}\right)\right|\right)\to 0, \end{aligned}$$

and in a similar way we have obtained

$$m\left|\triangle_{10}\left(\frac{b_{m-1,n}}{(m-1)n}\right)\right||F'_{m-1}(x)||D'_n(y)|\to 0,$$

$$n\left|\triangle_{01}\left(\frac{b_{m,n-1}}{m(n-1)}\right)\right||D'_m(x)||K'_{n-1}(y)|\to 0,$$

as $m+n\to\infty$.

Finally, using the estimate $|D'_p(v)|=O(p)$ we get

$$\frac{b_{m,n}}{mn}D'_m(x)D'_n(y)\to 0 \quad \text{as} \quad m+n\to\infty,$$

therefore

$$g(x,y)=\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}(k+1)(\ell+1)\triangle_{22}\left(\frac{b_{k,\ell}}{k\ell}\right)F'_k(x)K'_\ell(y),$$

exists, and it is integrable since

$$\begin{aligned} \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}(k+1)(\ell+1)\left|\triangle_{22}\left(\frac{b_{k,\ell}}{k\ell}\right)\right||F'_k(x)||K'_\ell(y)| \\ = O\left(\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}(k\ell)^2\left|\triangle_{22}\left(\frac{b_{k,\ell}}{k\ell}\right)\right|\right)<\infty. \end{aligned}$$

The proof is completed. ∎

Let $f(x,y)$ be a bivariate function, 2π-periodic in each variable. We say that the function $f(x,y)$ belongs to the class L_p, $1\le p<\infty$, if f is measurable and

$$\|f\|_p=\left\{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(x,y)|^p dxdy\right\}^{1/p}<+\infty.$$

We use the notation $\omega_{k_1,k_2}(f,t_1,t_2)_p$ for the mixed modulus of smoothness in L_p of order k_1 in x and k_2 in y of the function $f \in L_p$, i.e.

$$\omega_{k_1,k_2}(f,t_1,t_2)_p = \sup_{|h_1|\le t_1, |h_2|\le t_2} \|\triangle_{h_1 h_2}^{k_1 k_2} f(x,y)\|_p,$$

where

$$\triangle_{h_1 h_2}^{k_1 k_2} f(x,y) = \sum_{\mu=1}^{k_1}\sum_{\nu=1}^{k_2} (-1)^{k_1+k_2-\mu-\nu} \binom{k_1}{\mu}\binom{k_2}{\nu} f(x+\mu h_1, y+\nu h_2).$$

Now and below the letter C_{k_1,k_2} with or without subscripts denotes a constant having different values in different contexts and depending upon subscripts.

We establish the following result.

Theorem 2.3. *Let $\{b_{m,n}\} \in \mathcal{R}^2$. Then the following estimate holds:*

$$\begin{aligned}
&\omega_{k_1,k_2}\left(g, \frac{1}{m}, \frac{1}{n}\right)_1 \\
&\le C_{k_1,k_2}\Bigg\{ m^{-k_1} n^{-k_2} \sum_{k=1}^{m}\sum_{\ell=1}^{n} (k+1)^{k_1+2}(\ell+1)^{k_2+2} \left|\triangle_{22}\left(\frac{b_{k,\ell}}{k\ell}\right)\right| \\
&\qquad + m^{-k_1} \sum_{k=1}^{m}\sum_{\ell=n+1}^{\infty} (k+1)^{k_1+2}(\ell+1)^{2} \left|\triangle_{22}\left(\frac{b_{k,\ell}}{k\ell}\right)\right| \\
&\qquad + n^{-k_2} \sum_{k=m+1}^{\infty}\sum_{\ell=1}^{n} (k+1)^{2}(\ell+1)^{k_2+2} \left|\triangle_{22}\left(\frac{b_{k,\ell}}{k\ell}\right)\right| \\
&\qquad + \sum_{k=m+1}^{\infty}\sum_{\ell=n+1}^{\infty} (k+1)^{2}(\ell+1)^{2} \left|\triangle_{22}\left(\frac{b_{k,\ell}}{k\ell}\right)\right| \Bigg\}.
\end{aligned}$$

Proof. Theorem 2.2 implies that the function $g(x,y)$ is integrable. Also, we have seen that

$$g(x,y) = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} (k+1)(\ell+1)\triangle_{22}\left(\frac{b_{k,\ell}}{k\ell}\right) F_k'(x) K_\ell'(y).$$

The symmetry of the function f implies the equalities

$$|\triangle_{h_1 h_2}^{k_1 k_2} f(-x,y)| = |\triangle_{-h_1, h_2}^{k_1 k_2} f(x,y)|,$$

$$|\triangle_{h_1 h_2}^{k_1 k_2} f(x,-y)| = |\triangle_{h_1, -h_2}^{k_1 k_2} f(x,y)|,$$

and

$$|\triangle^{k_1k_2}_{h_1h_2}f(-x,-y)|=|\triangle^{k_1k_2}_{-h_1,-h_2}f(x,y)|.$$

Therefore we can write

$$\begin{aligned}&\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\triangle^{k_1k_2}_{h_1h_2}g(x,y)|dxdy\\&\qquad=\int_0^{\pi}\int_0^{\pi}|\triangle^{k_1k_2}_{-h_1,-h_2}g(x,y)|dxdy+\int_0^{\pi}\int_0^{\pi}|\triangle^{k_1k_2}_{-h_1,h_2}g(x,y)|dxdy\\&\qquad\quad+\int_0^{\pi}\int_0^{\pi}|\triangle^{k_1k_2}_{h_1,-h_2}g(x,y)|dxdy+\int_0^{\pi}\int_0^{\pi}|\triangle^{k_1k_2}_{h_1,h_2}g(x,y)|dxdy.\end{aligned}$$

Thus, to prove the theorem, it is sufficient to estimate

$$\mathcal{I}:=\int_0^{\pi}\int_0^{\pi}|\triangle^{k_1k_2}_{\pm h_1,\pm h_2}g(x,y)|dxdy\quad\text{for}\quad 0<h_1\le 1/m,\ 0<h_2\le 1/n.$$

Next, we write

$$\begin{aligned}\mathcal{I}&=\int_0^{\pi}\int_0^{\pi}\left|\triangle^{k_1k_2}_{\pm h_1,\pm h_2}\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}(k+1)(\ell+1)\triangle_{22}\left(\frac{b_{k,\ell}}{k\ell}\right)F'_k(x)K'_\ell(y)\right|dxdy\\&=\int_0^{(k_1+1)/m}\int_0^{(k_2+1)/n}+\int_0^{(k_1+1)/m}\int_{(k_2+1)/n}^{\pi}+\int_{(k_1+1)/m}^{\pi}\int_0^{(k_2+1)/n}\\&\quad+\int_{(k_1+1)/m}^{\pi}\int_{(k_2+1)/n}^{\pi}:=\sum_{s=1}^{4}\mathcal{I}_s.\end{aligned}$$

First for $\mathcal{I}_1$ we have

$$\begin{aligned}\mathcal{I}_1&\le\left(\sum_{k=1}^{m}\sum_{\ell=1}^{n}+\sum_{k=m+1}^{\infty}\sum_{\ell=1}^{n}+\sum_{k=1}^{m}\sum_{\ell=n+1}^{\infty}+\sum_{k=m+1}^{\infty}\sum_{\ell=n+1}^{\infty}\right)\Bigg[(k+1)(\ell+1)\\&\quad\times\left|\triangle_{22}\left(\frac{b_{k,\ell}}{k\ell}\right)\right|\int_0^{(k_1+1)/m}\left|\triangle^{k_1}_{\pm h_1}F'_k(x)\right|dx\int_0^{(k_2+1)/n}\left|\triangle^{k_2}_{\pm h_2}K'_\ell(y)\right|dy\Bigg]\\&:=\sum_{d=1}^{4}\mathcal{I}_1^{(d)}.\end{aligned}$$

Using Lemma 1.3 we obtain

$$\mathcal{I}_1^{(1)}\le C_{k_1,k_2}m^{-k_1}n^{-k_2}\sum_{k=1}^{m}\sum_{\ell=1}^{n}(k+1)^{k_1+2}(\ell+1)^{k_2+2}\left|\triangle_{22}\left(\frac{b_{k,\ell}}{k\ell}\right)\right|,$$

$$\mathcal{I}_1^{(2)} \leq C_{k_1,k_2} n^{-k_2} \sum_{k=m+1}^{\infty} \sum_{\ell=1}^{n} (k+1)^2 (\ell+1)^{k_2+2} \Big| \triangle_{22} \left(\frac{b_{k,\ell}}{k\ell} \right) \Big|,$$

$$\mathcal{I}_1^{(3)} \leq C_{k_1,k_2} m^{-k_1} \sum_{k=1}^{m} \sum_{\ell=n+1}^{\infty} (k+1)^{k_1+2} (\ell+1)^2 \Big| \triangle_{22} \left(\frac{b_{k,\ell}}{k\ell} \right) \Big|,$$

$$\mathcal{I}_1^{(4)} \leq C_{k_1,k_2} \sum_{k=m+1}^{\infty} \sum_{\ell=n+1}^{\infty} (k+1)^2 (\ell+1)^2 \Big| \triangle_{22} \left(\frac{b_{k,\ell}}{k\ell} \right) \Big|.$$

and therefore

$$\begin{aligned} \mathcal{I}_1 \leq C_{k_1,k_2} \Bigg\{ & m^{-k_1} n^{-k_2} \sum_{k=1}^{m} \sum_{\ell=1}^{n} (k+1)^{k_1+2} (\ell+1)^{k_2+2} \Big| \triangle_{22} \left(\frac{b_{k,\ell}}{k\ell} \right) \Big| \\ & + m^{-k_1} \sum_{k=1}^{m} \sum_{\ell=n+1}^{\infty} (k+1)^{k_1+2} (\ell+1)^2 \Big| \triangle_{22} \left(\frac{b_{k,\ell}}{k\ell} \right) \Big| \\ & + n^{-k_2} \sum_{k=m+1}^{\infty} \sum_{\ell=1}^{n} (k+1)^2 (\ell+1)^{k_2+2} \Big| \triangle_{22} \left(\frac{b_{k,\ell}}{k\ell} \right) \Big| \\ & + \sum_{k=m+1}^{\infty} \sum_{\ell=n+1}^{\infty} (k+1)^2 (\ell+1)^2 \Big| \triangle_{22} \left(\frac{b_{k,\ell}}{k\ell} \right) \Big| \Bigg\}. \end{aligned} \tag{6}$$

In a very similar way, using Lemma 1.3, it is verified that

$$\begin{aligned} \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4 \leq C_{k_1,k_2} \Bigg\{ & m^{-k_1} n^{-k_2} \sum_{k=1}^{m} \sum_{\ell=1}^{n} (k+1)^{k_1+2} (\ell+1)^{k_2+2} \Big| \triangle_{22} \left(\frac{b_{k,\ell}}{k\ell} \right) \Big| \\ & + m^{-k_1} \sum_{k=1}^{m} \sum_{\ell=n+1}^{\infty} (k+1)^{k_1+2} (\ell+1)^2 \Big| \triangle_{22} \left(\frac{b_{k,\ell}}{k\ell} \right) \Big| \\ & + n^{-k_2} \sum_{k=m+1}^{\infty} \sum_{\ell=1}^{n} (k+1)^2 (\ell+1)^{k_2+2} \Big| \triangle_{22} \left(\frac{b_{k,\ell}}{k\ell} \right) \Big| \\ & + \sum_{k=m+1}^{\infty} \sum_{\ell=n+1}^{\infty} (k+1)^2 (\ell+1)^2 \Big| \triangle_{22} \left(\frac{b_{k,\ell}}{k\ell} \right) \Big| \Bigg\}. \end{aligned} \tag{7}$$

Finally, combining (7) and (8), it follows that

$$\begin{aligned}
&\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\triangle_{h_1h_2}^{k_1k_2}g(x,y)|dxdy\\
&\le C_{k_1,k_2}\Bigg\{m^{-k_1}n^{-k_2}\sum_{k=1}^{m}\sum_{\ell=1}^{n}(k+1)^{k_1+2}(\ell+1)^{k_2+2}\Big|\triangle_{22}\left(\frac{b_{k,\ell}}{k\ell}\right)\Big|\\
&\qquad +m^{-k_1}\sum_{k=1}^{m}\sum_{\ell=n+1}^{\infty}(k+1)^{k_1+2}(\ell+1)^{2}\Big|\triangle_{22}\left(\frac{b_{k,\ell}}{k\ell}\right)\Big|\\
&\qquad +n^{-k_2}\sum_{k=m+1}^{\infty}\sum_{\ell=1}^{n}(k+1)^{2}(\ell+1)^{k_2+2}\Big|\triangle_{22}\left(\frac{b_{k,\ell}}{k\ell}\right)\Big|\\
&\qquad +\sum_{k=m+1}^{\infty}\sum_{\ell=n+1}^{\infty}(k+1)^{2}(\ell+1)^{2}\Big|\triangle_{22}\left(\frac{b_{k,\ell}}{k\ell}\right)\Big|\Bigg\}.
\end{aligned}$$

This completes the proof of the theorem. ∎

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Received: February 1, 2012